

Generalized Rabi models: diagonalization in the spin subspace and differential operators of Dunkl type

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Abstract – A discrete parity \mathbb{Z}_2 symmetry of a two parameter extension of the quantum Rabi model which smoothly interpolates between the latter and the Jaynes-Cummings model, and of the two-photon and the two-mode quantum Rabi models enables their diagonalization in the spin subspace. A more general statement is that the respective sets of 2×2 hermitian operators of the Fulton-Gouterman type and those diagonal in the spin subspace are unitary equivalent. The diagonalized representation makes it transparent that any question about integrability and solvability can be addressed only at the level of ordinary differential operators of Dunkl type. Braak's definition of integrability is shown (i) to contradict earlier numerical studies and (ii) to imply that any physically reasonable differential operator of Fulton-Gouterman type is integrable.

Introduction. – Any 2×2 hermitian operator \hat{H} can be expressed in the form

$$\hat{H} = \sum_{j=0}^3 h_j \sigma_j, \quad h_j = \frac{1}{2} \text{Tr} (\hat{H} \sigma_j), \quad (1)$$

where h_j 's are one-dimensional operators in a suitable Hilbert space \mathfrak{H} and here and elsewhere the standard representation of the Pauli matrices σ_j , $j = 1, 2, 3$, is assumed. For the sake of compactness, we set $\sigma_0 := \mathbb{1}$ in summation formulas, with $\mathbb{1}$ being the unit matrix. \hat{H} is said to be of the *Fulton-Gouterman* type [1], and denoted by \hat{H}_{FG} , if (i) \hat{H} is similar to

$$\hat{H}_{FG} = A\mathbb{1} + B\sigma_1 + C\sigma_2 + D\sigma_3, \quad (2)$$

and (ii) there is a hermitian operator \hat{R} such that

$$[\hat{R}, A] = [\hat{R}, B] = 0, \quad \{\hat{R}, C\} = \{\hat{R}, D\} = 0, \quad (3)$$

where $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ denote the conventional commutator and anticommutator. (Our definition of \hat{H}_{FG} is broader than the original one by including the term $C\sigma_2$ (cf. \hat{H} in eq. (4.1) of ref. [1]).)

A prominent example of the Fulton-Gouterman type Hamiltonians will be shown to be the generalized Rabi model (GRM) studied by Müller *et al.* [2, 3], Schiró *et al.* [4], Gritsev *et al.* [5, 6], and others [7],

$$\hat{H}_{gR} = \gamma a^\dagger a + \mu \sigma_3 + k_1 (a^\dagger \sigma_- + a \sigma_+) + k_2 (a^\dagger \sigma_+ + a \sigma_-), \quad (4)$$

and the two-photon (TPRM) and the two-mode quantum Rabi models (TMRM) [8] [cf. eq. (9) below]. Here \hat{a} and \hat{a}^\dagger are the conventional boson annihilation and creation operators satisfying commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. In the Fock-Bargmann representation [9, 10],

$$a \rightarrow d/dz = d_z, \quad a^\dagger \rightarrow z, \quad (5)$$

\hat{H}_{gR} becomes a first-order differential operator on $\mathfrak{B} \otimes \mathbb{C}^2$, where \mathfrak{B} is the Fock-Bargmann Hilbert space of entire analytic functions isomorphic to $L^2(\mathbb{R})$ [9]. The GRM interpolates between the Jaynes-Cummings model (JCM) [11] (for $k_2 = 0$) and the original quantum Rabi model (RM) [12–22] (for $k_1 = k_2 = k$). The RM describes the simplest interaction between a cavity mode with a bare frequency ω and a two-level system, or a qubit, with a bare resonance frequency ω_0 . One has $\gamma = \hbar\omega > 0$, $k_1 = k_2 = \hbar g > 0$, with g being a coupling constant, $\mu = \hbar\omega_0/2$, where \hbar is the reduced Planck constant. The RM with a *negative* sign of its parameters g and μ (cf. eq. (12) of ref. [14]) is used to describe an excitation hopping between two sites (μ is then a tunneling parameter) and is relevant in understanding the transition between untrapped and trapped behavior of an exciton. The GRM serves as a non-trivial model in spin resonance, for various problems involving the interaction between electronic and vibrational degrees of freedom in molecules and solids, and in quantum optics [23–27]. The RM and GRM are

presently the focus of intense experimental and theoretical activity for cavity- and circuit-QED setups, superconducting q-bits, nitrogen vacancy centers, etc. [6, 23–27]. With new experiments rapidly approaching the limit of the so-called deep strong coupling regime, there is every reason to believe that such systems could open up a rich vein of research on truly quantum effects with implications for quantum information science and fundamental quantum optics. There are several further motivations to consider this model. It can be mapped onto the model describing a two-dimensional electron gas with Rashba ($\alpha_R \sim k_1$) and Dresselhaus ($\alpha_D \sim k_2$) spin-orbit couplings subject to a perpendicular magnetic field (the Zeeman splitting thereby equals 2μ) [5, 6, 28]. The Rashba spin-orbit coupling (SOC) can be tuned by an applied electric field while the Zeeman term is tuned by an applied magnetic field. This allows us to explore the whole parameter space of the model. In ref. [29] a possible realization of tunable Rashba and Dresselhaus SOC with ultracold alkali atoms is proposed, where each state is coupled by a two-photon Raman transition. Further examples of physical realizations of the GRM include (i) electric-magnetic coupling of light and matter, and (ii) effective realization of the model using 3- and 4-level emitters [6].

Provided that $\hat{R}^2 = 1$, any \hat{H}_{FG} enjoys a discrete \mathbb{Z}_2 -symmetry $\hat{\Pi}_{FG} = \hat{R}\sigma_1$: $[\hat{H}_{FG}, \hat{\Pi}_{FG}] = 0$, $\hat{\Pi}_{FG}^2 = 1$. The \mathbb{Z}_2 -symmetry suffices to partially diagonalize \hat{H}_{FG} operating on the Hilbert space $\mathfrak{B} \otimes \mathbb{C}^2$ in the spin subspace [30]. Indeed, a sufficient condition for the spin-subspace diagonalization of a Hamiltonian \hat{H} on $\mathfrak{B} \otimes \mathbb{C}^N$ is that \hat{H} possesses an *Abelian* symmetry G of the order N . Surprisingly enough, the diagonalization of the GRM, TPRM, and TMRM in the spin subspace has not been discussed yet - *cf.* refs. [2–8, 12, 13, 15, 17–22] - even though the diagonalization can be performed by rather straightforward unitary transformation:

Theorem 1: Any \hat{H}_{FG} given by (2) can be diagonalized in the spin subspace by means of a unitary transformation,

$$\begin{aligned} U_{FG} \hat{H}_{FG} U_{FG}^{-1} &= (A + D)\mathbb{1} + B\hat{R}\sigma_3 - iC\hat{R}\sigma_3, \\ U_{FG} \hat{\Pi}_{FG} U_{FG}^{-1} &= \sigma_3, \end{aligned} \quad (6)$$

induced by

$$U_{FG} = \frac{1}{2} \left[(1 + \hat{R})U_{13} + (1 - \hat{R})U_2^{-1} \right], \quad (7)$$

where $U_{13} = (\sigma_1 + \sigma_3)/\sqrt{2}$ and $U_2 = (\mathbb{1} + i\sigma_2)/\sqrt{2}$. Thereby an original spin-1/2 problem in the Hilbert space $\mathfrak{H} \otimes \mathbb{C}^2$ decouples into two distinct one-dimensional problems in \mathfrak{H} , each characterized by the operator

$$\hat{L}_{\pm} = A + D \pm (B - iC)\hat{R}, \quad (8)$$

where the \pm sign corresponds to the respective positive and negative parity eigenspaces. \square

The letter is organized as follows. Theorem 1 is applied to the GRM, TPRM and TMRM. For the GRM the oper-

ators \hat{L}_{\pm} become first-order ordinary differential operators of *Dunkl* type [31, 32], whereas for the TPRM and TMRM they become second-order differential operators of Dunkl type. The Dunkl type operators, which are characterized in that they contain a *reflection* operator, became a branch of mathematics only as late as 1990 [31]. Working in the diagonalized representation makes it transparent that any question about integrability and solvability can be addressed only at the level of ordinary differential operators of Dunkl type. Braak's definition of integrability [18] is shown (i) to imply that any physically reasonable differential operator of Dunkl type is integrable and (ii) to contradict earlier numerical studies by Müller *et al.* [2, 3].

For completeness an adaption of Wagner's proof directly to \hat{H}_{FG} as given by (2) is provided. The reverse of Wagner's theorem is also shown to be true and the following result is proven:

Theorem 2: A hermitian operator \hat{H} has the Fulton-Gouterman form (2), (3) if and only if \hat{H} is unitary equivalent to an operator diagonal in the spin subspace. \square

Diagonalization in the spin subspace. – In view of Theorem 1, it suffices to demonstrate the Fulton-Gouterman form of a given Hamiltonian. \hat{H}_{gR} can be brought into the Fulton-Gouterman form (2) upon applying a (nonunitary) similarity transformation $\hat{\mathcal{H}}_{gR} = \mathbf{W}\hat{H}_{gR}\mathbf{W}^{-1}$ with

$$\mathbf{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} w & w^{-1} \\ w & -w^{-1} \end{pmatrix}, \quad \mathbf{W}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} w^{-1} & w^{-1} \\ w & -w \end{pmatrix},$$

where $w = (k_2/k_1)^{1/4}$. Under the similarity transformation

$$\begin{aligned} \sigma_+ &\rightarrow \frac{w^2}{2}(\sigma_3 - i\sigma_2), \quad \sigma_- \rightarrow \frac{1}{2w^2}(\sigma_3 + i\sigma_2), \\ k_1\sigma_+ + k_2\sigma_- &\rightarrow \sqrt{k_1 k_2}\sigma_3, \quad \sigma_3 \rightarrow \sigma_1, \\ k_2\sigma_+ + k_1\sigma_- &\rightarrow \frac{k_1^2 + k_2^2}{\sqrt{k_1 k_2}}\sigma_3 + i\frac{k_1^2 - k_2^2}{\sqrt{k_1 k_2}}\sigma_2. \end{aligned}$$

The transformed $\hat{\mathcal{H}}_{gR}$ divided by γ takes on the Fulton-Gouterman form (2) with

$$A = \hat{a}^\dagger \hat{a}, \quad B = \Delta, \quad C = i\frac{\lambda_-}{\kappa} \hat{a}^\dagger, \quad D = \kappa a + \frac{\lambda_+}{\kappa} \hat{a}^\dagger,$$

where

$$\Delta := \frac{\mu}{\gamma}, \quad \lambda_{\pm} := \frac{k_1^2 \pm k_2^2}{2\gamma^2}, \quad \kappa := \frac{\sqrt{k_1 k_2}}{\gamma}.$$

The Fulton-Gouterman symmetry operation is realized by unitary $\hat{R} = e^{i\pi \hat{a}^\dagger \hat{a}}$, which induces *reflections* of the annihilation and creation operators: $\hat{a} \rightarrow -\hat{a}$, $\hat{a}^\dagger \rightarrow -\hat{a}^\dagger$, and leaves the boson number operator $\hat{a}^\dagger \hat{a}$ invariant [1]. According to Wagner's theorem, $\hat{\mathcal{H}}_{gR}$ in the Fock-Bargmann representation (5) is unitary equivalent to

$$\hat{\mathcal{H}}_{gR} = \left[(z + \kappa)d_z + \frac{\lambda_+ z}{\kappa} \right] + \left(\frac{\lambda_- z}{\kappa} + \Delta \right) \sigma_3 \hat{R}.$$

In the limit $k_1 = k_2 = g$ leading to the RM in a unitary equivalent *single-mode spin-boson* picture:

$$\lambda_- = 0, \quad \kappa = \frac{g}{\omega}, \quad \frac{\lambda_+}{\kappa} = \kappa,$$

$$A = z d_z, \quad B = \Delta, \quad C = 0, \quad D = \kappa(z + d_z),$$

(we set the reduced Planck constant $\hbar = 1$) and \mathbf{W} becomes the unitary transformation U_{13} [15, 20]:

$$\mathbf{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (\sigma_1 + \sigma_3) \equiv U_{13} = U_{13}^{-1}.$$

(A different \mathbf{W} has been used by Gritsev *et al.* [6] which in its symmetrized form reduces to $(\sigma_1 - \sigma_3)/\sqrt{2}$ in the limit $k_1 = k_2$.)

The Hamiltonians $\hat{H}_{2p} = \omega a^\dagger a + \beta \sigma_3 + g \sigma_1 [(a^\dagger)^2 + a^2]$ and $\hat{H}_{2m} = \omega(a_1^\dagger a_1 + a_2^\dagger a_2) + \beta \sigma_3 + g \sigma_1(a_1^\dagger a_2^\dagger + a_1 a_2)$ of the TPRM and the TMRM, respectively, are on unitary transforming with U_{13} brought into the Foulton-Gouterman form (*cf.* eqs. (4.3) and (5.3) of [8])

$$\hat{H}_{2FG} = \gamma(K_0 - c) + \Delta \sigma_1 + \sigma_3(K_+ + K_-), \quad (9)$$

where $\gamma = \omega/g$, $\Delta = \beta/g$. Compared to the *Heisenberg* algebra of a , a^\dagger in \hat{H}_{gR} , the operators K_\pm, K_0 in (9) form the usual $su(1, 1)$ Lie algebra, $[K_0, K_\pm] = \pm K_\pm$, $[K_+, K_-] = -2K_0$. In the case of the TPRM, $c = 1/4$,

$$K_+ = \frac{1}{2}(a^\dagger)^2, \quad K_- = \frac{1}{2}a^2, \quad K_0 = \frac{1}{2}(a^\dagger a + \frac{1}{2}),$$

whereas, in the case of the TMRM, $c = 1/2$, and

$$K_+ = a_1^\dagger a_2^\dagger, \quad K_- = a_1 a_2, \quad K_0 = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1).$$

The parity symmetry $\hat{\Pi}_{FG} = \hat{R} \sigma_1$ is realized by unitary $\hat{R} = e^{i\pi K_0}$, which induces reflections of K_\pm and leaves K_0 invariant.

The Fock-Bargmann Hilbert space \mathfrak{B} is based on the coherent states associated with the Heisenberg algebra [10]. In the present case, \mathfrak{B} gets replaced by a more general Hilbert space of entire analytic functions of growth (1, 1) associated with the so-called Barut-Girardello coherent states [10] of the annihilation operator K_- of the $su(1, 1)$ Lie algebra. In an infinite-dimensional unitary irreducible representation, known as the positive discrete series $\mathcal{D}^+(q)$, the operators K_\pm, K_0 of the TPRM, which realize the single-mode bosonic representation of $su(1, 1)$, are represented as differential operators,

$$K_0 = z d_z + q, \quad K_+ = z/2, \quad K_- = 2z d_z^2 + 4q d_z,$$

where the parameter q , called Bargmann index, satisfies $q = 1/4, 3/4$. The operators K_\pm, K_0 of the TMRM providing the two-mode bosonic representation of $su(1, 1)$ have the single-variable differential realization as

$$K_0 = z d_z + q, \quad K_+ = z, \quad K_- = z d_z^2 + 2q d_z,$$

where $q > 0$ can be any integer or half-integer [8, 10].

Linear differential operators of Dunkl type. –

The action of \hat{R} in the above cases reduces to reflections $\hat{R}f(z) = f(-z)$ in a suitable Hilbert space of entire analytic functions. In the case of the GRM, the respective diagonal components \hat{L}_\pm defined by (8) become linear first-order ordinary differential operators of Dunkl type,

$$\hat{L}_\pm = (z + \kappa) d_z + \frac{\lambda_\pm z}{\kappa} \pm \left(\frac{\lambda_- z}{\kappa} + \Delta \right) \hat{R}. \quad (10)$$

In the limit of the RM one recovers

$$\hat{L}_\pm = (z + \kappa) d_z + \kappa z \pm \Delta \hat{R} \quad (11)$$

(*cf.* eq. (21) of ref. [14] and eq. (2.1) of ref. [16]). For the respective TPRM and the TMRM one finds

$$\begin{aligned} \hat{L}_{\pm;2p} &= 2z d_z^2 + (4q + \gamma z) d_z + \frac{z}{2} + \gamma \left(q - \frac{1}{4} \right) \pm \Delta \hat{R}, \\ \hat{L}_{\pm;2m} &= z d_z^2 + (2q + \gamma z) d_z + z + \gamma \left(q - \frac{1}{2} \right) \pm \Delta \hat{R}. \end{aligned} \quad (12)$$

For a general \hat{L}_\pm in (8) one has

$$[\hat{L}_\pm, \hat{R}] = \mp 2iC + 2D\hat{R} \neq 0.$$

Therefore also $[\hat{L}_\pm, \hat{R}\hat{L}_\pm] \neq 0$. Whereas for an eigenvector ϕ of \hat{L}_\pm one has $\hat{R}\hat{L}_\pm\phi = \epsilon\hat{R}\phi$, one cannot say anything definite about $\hat{L}_\pm\hat{R}\phi$.

Note that in the absence of the reflection operator \hat{R} in eq. (11), *e.g.* with $\hat{L}_\pm = (z + \kappa) d_z + \kappa z \pm \Delta$, the eigenvalue problem,

$$(\hat{L} - \epsilon)\phi = 0, \quad (13)$$

can be easily integrated. One finds $\phi(z) = \text{const}(z + \kappa)^{\kappa^2 \pm \Delta - \epsilon} e^{-\kappa z}$, where ‘const’ is an integration constant. The solutions will be holomorphic if and only if $\kappa^2 \pm \Delta - \epsilon \in \mathbb{N}_0$. In spite of a deceptive simplicity of \hat{L}_\pm in eqs. (10), (11), a rigorous analytic solution of (13) remains an unsolved problem (in the sense that analytic expressions for eigenvalues are not known - *cf.* ref. [33]). This demonstrates that the reflection operator \hat{R} is a highly nontrivial obstruction for solving the eigenvalue problem (13) [32].

Each term of the one-dimensional operator \hat{L}_\pm in (10), (11), (12) does not change the degree of a monomial z^n by more than ± 1 . Thereby the resulting eigenvalue problem (13) naturally reduces to a *three-term recurrence relation* (TTRR) [8, 13, 17, 19, 20, 34]. Thus each \hat{L}_\pm corresponds to the so-called irreducible component of ref. [21]. Those have been shown to have a *nondegenerate* spectrum under very general conditions, and hence no level crossing while varying coupling parameter(s). Alternatively, the nondegeneracy applies to all problems where the Hamiltonian operator is a self-adjoint extension of a *tridiagonal* Jacobi matrix of deficiency index (1, 1) [17, 35]. Therefore, all \hat{H}_{FG} leading to a TTRR have *avoided* level crossings. The above arguments provide rigorous proof for the avoided crossing observed numerically by Müller *et al.* [2, 3].

Braak's definition of integrability. — According to Braak's definition of integrability [18]: If each eigenstate of a quantum system with f_1 discrete and f_2 continuous degrees of freedom can be *uniquely* labeled by $f_1 + f_2 = f$ quantum numbers $\{d_1, \dots, d_{f_1}, c_1, \dots, c_{f_2}\}$, such that the d_j can take on $\dim(H_j)$ different values, where H_j is the state space of the j th discrete degree of freedom and the c_k range from 0 to infinity, then this system is *quantum integrable*. The RM has $f_1 = f_2 = 1$ and degeneracies take place only between levels of states with *different* parity, whereas within the parity subspaces no level crossings occur. The global label for the RM [valid for all values of κ in eq. (11)] is two dimensional, with one label for the parity and the other being the energy sorting number within a given parity subspace, and hence, according to Braak, the RM is quantum *integrable*. But this leads to an inflation of integrable models, because the *avoided* level crossings between states of equal parity is generic for the models studied here. Following Braak's arguments, all physically reasonable \hat{H}_{FG} are necessarily quantum integrable.

Braak's arguments appear to be based on a wrong assessment of the role of *discrete* symmetries. The latter divide the Hilbert space of \hat{H}_{FG} into invariant subspaces. In general, this does not result in symmetry-induced level-degeneracies, but it does lead to accidental degeneracies between levels belonging to different invariant subspaces (*cf.* Judd solutions). Such level crossings exist independently of whether or not \hat{H}_{FG} is integrable [3].

The general rule has always been to analyze corresponding invariant or irreducible components. For instance, in statistical analysis of the eigenvalues of quantum billiards one performs the so-called *desymmetrization*, which reduces the study to a fundamental domain of a discrete group [36]. The diagonalized representation makes it transparent that integrability and solvability of the GRM can be addressed only at the level of first-order one-dimensional differential operators \hat{L}_\pm . The latter necessitates considering each invariant parity subspace independently. In their thorough numerical studies, Müller *et al.* [2, 3] did just that. They made use of the fact that a quantum integrability cannot be inferred from *quantum invariants* as simply as classical integrability can be inferred from integrals of the motion (*analytic invariants*). Commuting operators can always be constructed *irrespective* of whether the model is (classically) integrable or not [2, 3, 37]. When Einstein-Brillouin-Keller quantization is possible, it applies to all conserved dynamical variables (not only to the Hamiltonian) and in particular to the *time average* of *any* dynamical variable. Any operator T that is not already an invariant, $[H, T] \neq 0$, can be turned into an invariant via time average. In the *energy* representation, the time average strips T of all its off-diagonal elements. The resulting operator $I_T = \langle T \rangle$ being *diagonal* in the energy representation thus commutes with H by construction [37]. Müller *et al.* [2, 3] studied two-dimensional patterns of quantum invariants $\{(\epsilon_n, \langle T_j \rangle_n)\}$, where ϵ_n is the n th eigenvalue, and $T_1 = a^\dagger \sigma_+$, $T_2 = a^\dagger (\sigma_- + \sigma_+)$.

(Although T_j 's are not hermitian, the matrix elements $\langle n | T_j | n \rangle = \langle T_j \rangle_n$ happen to be real for all energy eigenstates [2].) The patterns of points $\{(\epsilon_n, \langle T_j \rangle_n)\}$ in invariant parity subspaces were found to be strikingly different for the respective integrable ($k_1 k_2 = 0$) and nonintegrable ($k_1 k_2 \neq 0$) cases. A *qualitative* change in pattern required the assignment of mutually exclusive sets of quantum numbers to the same set of eigenstates in different parameter regimes [2, 3]. In the integrable cases, the patterns formed two separate linear strands of points. Level crossings required a two dimensional label for an unambiguous assignment of levels, each label corresponding to one of the respective quantum invariants. Contrary to that, a single *level sorting* quantum number sufficed to label all eigenstates in the presence of the *avoided* level crossings between states of equal parity for nonintegrable cases. In contrast to Braak's conclusion, avoided level crossings were found to be the trademark of quantum *nonintegrability*. The integrable and nonintegrable cases revealed also unambiguously different patterns of coordinated motion of all states with given parity in the plane of invariants $(\epsilon_n, \langle T_2 \rangle_n)$ as the interaction strength (*i.e.* Λ in the parametrization $k_1 = \Lambda \cos \alpha$, $k_2 = \Lambda \sin \alpha$ of the coupling constants) gradually increased [3]. The distinctive attributes of quantum invariants in the integrable and nonintegrable regimes of a quantum system are subtle but *not* ambiguous. As soon as $k_1 k_2 \neq 0$ (or $\alpha \neq 0, \pi/2$), the GRM was found *nonintegrable* [2, 3].

Proof of Theorem 1. — It is expedient to introduce

$$U_{jkl} = \frac{1}{2} \left[(1 + \hat{R}) U_{jk} + (1 - \hat{R}) U_l \right] \quad (14)$$

with unequal $j, k, l = 1, 2, 3$, where

$$U_{jk} = (\sigma_j + \sigma_k)/\sqrt{2}, \quad U_l = (\mathbf{1} + i\sigma_l)/\sqrt{2}, \quad (15)$$

are 2×2 unitary matrices. Any U_{jkl} is thus a linear combination of unitary matrices with the coefficients being one-dimensional projectors $\hat{P}_\pm = (1 \pm \hat{R})/2$. U_{jkl} itself is unitary:

$$\begin{aligned} U_{jkl} U_{jkl}^{-1} &= \frac{1}{8} \left[(1 + \hat{R}) U_{jk} + (1 - \hat{R}) U_l \right] \\ &\quad \times \left[(1 + \hat{R}) U_{jk} + (1 - \hat{R}) U_l^{-1} \right] \\ &= \frac{1}{4} \left[(1 + \hat{R})^2 + (1 - \hat{R})^2 \right] \mathbf{1} \\ &= \frac{1}{4} \left[2 + 2\hat{R} + 2 - 2\hat{R} \right] \mathbf{1} = \mathbf{1}. \end{aligned}$$

Now,

$$\begin{aligned} U_{jkl} \sigma_j U_{jkl}^{-1} &= \frac{1}{4} \left[(1 + \hat{R})^2 U_{jk} \sigma_j U_{jk} + (1 - \hat{R})^2 U_l \sigma_j U_l^{-1} \right]. \end{aligned}$$

For unequal $j, k, l = 1, 2, 3$ one finds

$$\begin{aligned} U_{jk} \sigma_j U_{jk} &= \sigma_k, \quad U_{jk} \sigma_l = -\sigma_l U_{jk}, \quad U_l \sigma_j = \sigma_j U_l^{-1}, \\ U_l \sigma_j U_l^{-1} &= -\epsilon_{ljk} \sigma_k, \quad U_l^{-1} \sigma_j U_l = \epsilon_{ljk} \sigma_k, \end{aligned} \quad (16)$$

where ϵ_{ljk} is the usual totally antisymmetric Levi-Civita symbol. In arriving at the final results we have repeatedly used

$$-i\sigma_1\sigma_2\sigma_3 = \mathbb{1}, \quad \sigma_j\sigma_k = i\epsilon_{jkl}\sigma_l. \quad (17)$$

Under the action of unitary U_l the matrix σ_l remains invariant. It holds trivially that $U_{jkl}\sigma_0U_{jkl}^{-1} = \sigma_0$. Given the properties (16), one can verify that (modulo a sign change and multiplication by \hat{R}):

- (*) the unitary transformation induced by U_{jkl} with unequal $j, k, l = 1, 2, 3$ interchanges σ_j and σ_k while leaving σ_0 and σ_l invariant.

For any operator \hat{X} on \mathfrak{H} commuting with \hat{R}

$$U_{jkl}\hat{X}\sigma_jU_{jkl}^{-1} = \hat{X}(U_{jkl}\sigma_jU_{jkl}^{-1}), \quad (18)$$

whereas for any operator \hat{Y} on \mathfrak{H} anticommuting with \hat{R}

$$U_{jkl}\hat{Y}\sigma_jU_{jkl}^{-1} = (U_{jkl}\hat{Y}U_{jkl}^{-1})(U_{jkl}\sigma_jU_{jkl}^{-1}). \quad (19)$$

Table 1. $U_{FG} = \frac{1}{2}[(1 + \hat{R})U_{13} + (1 - \hat{R})U_2^{-1}]$.

\hat{O}	$A\mathbb{1}$	$B\sigma_1$	$C\sigma_2$	$D\sigma_3$
$U_{FG}\hat{O}U_{FG}^{-1}$	$A\mathbb{1}$	$B\hat{R}\sigma_3$	$-iC\hat{R}\sigma_3$	$D\mathbb{1}$

The unitary transformation U_{FG} of Theorem 1 is a particular case of U_{jkl} defined by (14). One has $U_{FG} = U_{132}^{-1} = U_{13,-2}$, where the minus sign in front of 2 stands for the inverse of U_2 in the definition (7). On combining relations (16),

$$U_{FG}\sigma_1U_{FG}^{-1} = \frac{1}{4}\sigma_3[(1 + \hat{R})^2 - (1 - \hat{R})^2] = \hat{R}\sigma_3.$$

Hence for A and B commuting with \hat{R} one has $U_{FG}A\sigma_0U_{FG}^{-1} = A\sigma_0$ and $U_{FG}B\sigma_1U_{FG}^{-1} = B\hat{R}\sigma_3$, respectively. With the help of identities (16) one has

$$\begin{aligned} U_{FG}\sigma_0U_{FG}^{-1} &= \sigma_0, & U_{FG}\sigma_1U_{FG}^{-1} &= \hat{R}\sigma_3, \\ U_{FG}\sigma_2U_{FG}^{-1} &= -\hat{R}\sigma_2, & U_{FG}\sigma_3U_{FG}^{-1} &= \sigma_1, \end{aligned} \quad (20)$$

conforming to the general rule (*). Because

$$U_{jk}U_l = U_l^{-1}U_{jk} = \frac{1}{2}[\sigma_j + \sigma_k - \epsilon_{jkl}(\sigma_j - \sigma_k)],$$

i.e., $U_{13}U_2 = U_2^{-1}U_{13} = \sigma_1$, one has

$$\begin{aligned} U_{FG}\hat{Y}U_{FG}^{-1} &= \frac{1}{4}\hat{Y}[(1 - \hat{R})^2U_{13}U_2 + (1 + \hat{R})^2U_2^{-1}U_{13}] \\ &= \frac{1}{4}\hat{Y}\sigma_1[(1 - \hat{R})^2 + (1 + \hat{R})^2] = \hat{Y}\sigma_1. \end{aligned} \quad (21)$$

Eventually, on combining (19), (20), and (21):

$$\begin{aligned} U_{FG}C\sigma_2U_{FG}^{-1} &= C\sigma_1U_{FG}\sigma_2U_{FG}^{-1} = -iC\hat{R}\sigma_3, \\ U_{FG}D\sigma_3U_{FG}^{-1} &= D\sigma_1U_{FG}\sigma_3U_{FG}^{-1} = D\mathbb{1}. \end{aligned} \quad (22)$$

Therefore, the action of U_{FG} summarized in Table 1 ensures that any \hat{H}_{FG} of the Fulton-Gouterman type defined by (2), (3) can indeed be diagonalized in the spin subspace. The form of unitary transformed $\hat{\Pi}_{FG}$ and of operators \hat{L}_\pm in (8) can be read off from Table 1. Thereby the proof is completed.

Proof of Theorem 2. — If a hermitian operator \hat{H} has the Fulton-Gouterman form (2), (3), then, according to Theorem 1, it is unitary equivalent to an operator diagonal in the spin subspace. Hence in order to prove Theorem 2 it suffices to show that the reverse holds, too.

A hermitian operator \hat{H} is diagonal in the spin subspace if and only if $h_1 = h_2 \equiv 0$ in the expansion (1). Now any $h_j \neq 0$ in (1) can be decomposed as $h_j = \hat{X}_j + \hat{Y}_j$, where $[\hat{X}_j, \hat{R}] = 0$ and $\{\hat{Y}_j, \hat{R}\} = 0$, with \hat{R} being an arbitrary reflection operator. To this end, one takes

$$\hat{X}_j = \frac{1}{2}(h_j + \hat{R}h_j\hat{R}), \quad \hat{Y}_j = \frac{1}{2}(h_j - \hat{R}h_j\hat{R}).$$

A unitary U which commutes with any \hat{X}_j , $j = 0, 3$, and brings a diagonal operator \hat{H} into the Fulton-Gouterman form has to necessarily satisfy

$$U\sigma_3U^{-1} = \sigma_1. \quad (23)$$

At the same time, the transformed set $\{U\hat{Y}_0\sigma_0U^{-1}, U\hat{Y}_3\sigma_3U^{-1}\}$ has to become $\{\hat{Y}'\sigma_2, \hat{Y}''\sigma_3\}$, where the set $\{\hat{Y}', \hat{Y}''\}$ is, modulo a possible sign change and multiplication by \hat{R} and a constant, equivalent to $\{\hat{Y}_0, \hat{Y}_3\}$. In conformity to the general rule (*), the condition (23) fixes U_{jkl} to be either $U_{FG} = U_{132}^{-1}$ or U_{132} . The first choice can be excluded in virtue of the second of eqs. (22). In the case of U_{132} , one finds with the help of identities (16)

$$\begin{aligned} U_{132}\sigma_0U_{132}^{-1} &= \sigma_0, & U_{132}\sigma_1U_{132}^{-1} &= \sigma_3, \\ U_{132}\sigma_2U_{132}^{-1} &= -\hat{R}\sigma_2, & U_{132}\sigma_3U_{132}^{-1} &= \hat{R}\sigma_1, \end{aligned} \quad (24)$$

which is consistent with eqs. (20). Because U_{jk} is symmetric in its indices, one can always adopt the convention that, when calculating the products $U_{jk}U_l^{-1} = U_lU_{jk}$ with unequal j, k, l , the indices are ordered such that $\epsilon_{jkl} = 1$. With the above convention

$$U_{jk}U_l^{-1} = U_lU_{jk} = \sigma_j,$$

i.e., $U_{31}U_2^{-1} = U_2U_{31} = \sigma_3$, and one finds [cf. eq. (21)]

$$U_{132}\hat{Y}U_{132}^{-1} = \hat{Y}\sigma_3. \quad (25)$$

Eventually, in virtue of identities (19), (24), (25),

$$\begin{aligned} U_{132}\hat{Y}\sigma_0U_{132}^{-1} &= \hat{Y}\sigma_3, & U_{132}\hat{Y}\sigma_1U_{132}^{-1} &= \hat{Y}\mathbb{1}, \\ U_{132}\hat{Y}\sigma_2U_{132}^{-1} &= i\hat{Y}\hat{R}\sigma_1, & U_{132}\hat{Y}\sigma_3U_{132}^{-1} &= i\hat{Y}\hat{R}\sigma_2. \end{aligned}$$

Therefore, the unitary transformation induced by U_{132} transforms the set $\{(\hat{X}_0 + \hat{Y}_0)\sigma_0, (\hat{X}_3 + \hat{Y}_3)\sigma_3\}$ into $\{\hat{X}_0\sigma_0, \hat{X}_3\hat{R}\sigma_1, i\hat{Y}_3\hat{R}\sigma_2, \hat{Y}_0\sigma_3\}$, thereby yielding the Fulton-Gouterman form (2), (3). The proof is completed.

Conclusions. — The respective sets of 2×2 hermitian operators of the Fulton-Gouterman type and those diagonal in the spin subspace were shown to be unitary equivalent. As an example, discrete parity \mathbb{Z}_2 symmetry of a two

parameter extension of the Rabi model which smoothly interpolates between the latter and the Jaynes-Cummings model, the so-called generalized Rabi model (GRM), and of the two-photon and the two-mode quantum Rabi models was shown to enable their diagonalization in the spin subspace. The demonstrated diagonalized representation is expected to greatly simplify the description of time evolution and dissipative dynamics of the models. In the case of the GRM, supersymmetry on certain submanifolds in a parameter space has been established by Gritsev *et al.* [6]. The diagonalized representation could facilitate here a much straightforward identification of supercharges by halving the dimensions of matrices involved.

An intimate relation of the generalized Rabi models with the class of differential operators of Dunkl type was established. Hopefully, this will help to address computational issues more efficiently. Many problems involving parity symmetry appear as potential candidates of further examples where one could encounter the Dunkl type operators. The diagonalization can be straightforwardly extended to spin $s > 1/2$ models which possess an Abelian symmetry of the order of $N = 2s + 1$ [30, 38]. However the relation with the Dunkl type operators seems to be particular for spin $s = 1/2$ models: for $N > 2$ the Dunkl type operators are associated, in general, to nonabelian Coxeter groups [31].

The well known level-statistics criteria which have been applied with great success to autonomous particle systems are not applicable to the generalized Rabi models. The nearest-neighbour distribution of levels is not of the general type associated with chaotic systems and does not offer any conclusive evidence for quantum nonintegrability [39]. Only the analysis of two-dimensional patterns of quantum invariants $\{(\epsilon_n, \langle T \rangle_n)\}$ yields an unambiguous answer here. Braak's definition of integrability was shown not only to contradict the earlier pattern studies by Müller *et al.* [2, 3] but also to imply that any physically reasonable differential operator of Fulton-Gouterman type (*i.e.* leading to a TTRR) is integrable. This suggests that Braak's definition of integrability is most probably a faulty one. This is supported by the conclusions of ref. [40] that the Rabi model is *not* Yang-Baxter integrable.

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